

Space Inversion of Spinors Revisited: A Possible Explanation of Chiral Behavior in Weak Interactions

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Abstract

We investigate a model in which spinors are considered as being embedded within the Clifford algebra that operates on them. In Minkowski space $M_{1,3}$, we have four independent 4-component spinors, each living in a different minimal left ideal of $Cl(1,3)$. We show that under space inversion, a spinor of one left ideal transforms into a spinor of another left ideal. This brings novel insight to the role of chirality in weak interactions. We demonstrate the latter role by considering an action for a generalized spinor field $\psi^{\alpha i}$ that has not only a spinor index α but also an extra index i running over four ideals. The covariant derivative of $\psi^{\alpha i}$ contains the generalized spin connection, the extra components of which are interpreted as the $SU(2)$ gauge fields of weak interactions and their generalization. We thus arrive at a system that is left-right symmetric due to the presence of a “parallel sector”, postulated a long time ago, that contains mirror particles coupled to mirror $SU(2)$ gauge fields.

Key words: Mirror symmetry, mirror particles, weak interactions, algebraic spinors, Clifford algebras

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1 Introduction

After the seminal paper by Lee and Yang [1], we have become accustomed to the idea that the physical processes associated with weak interactions are not invariant under space inversion. However, in this same paper, Lee and Yang proposed that the introduction of mirror particles restores the invariance of β -decay under space inversion. The idea of mirror particles and the exact parity model has been thereafter pursued and investigated from various points of view in Refs. [2]–[5]. The possibility that mirror particles are responsible for dark matter has been explored in many works [6]. For a review on mirror particles and their phenomenological consequences, see [7].

In this paper, we will show how mirror particles find a natural explanation if we employ the concept of algebraic spinors. It is well known that spinors can be described as elements of the left or right minimal ideals of a Clifford algebra [8, 9]. As observed in Ref. [10], such an approach yields a more general and coherent theory than the traditional “column” representation. It incorporates the usual spinor formalism, but in addition, it brings the features that are absent from the usual theory in which spin space is a representation space lying outside of the algebra that operates on it. In a more general theory, a spinor can be considered as being embedded within the Clifford algebra that operates on it. This opens a Pandora’s box of possibilities that have been explored in the attempts to find a unified theory of fundamental particles and forces [11]–[13].

We will follow the approach [16, 17] in which spinors are constructed in terms of nilpotents formed from the spacetime basis vectors represented as generators of the Clifford algebra $Cl(1, 3)$. We find that under space inversion, a left-handed spinor of one ideal transforms into a right-handed spinor of not the same but of a different ideal. We then construct a specific model starting from a generalized Dirac action that is a functional of 16 complex valued fields, components of a generalized algebraic spinor field. Such a 16-component field is written as the 4×4 matrix $\psi^{\alpha i}$, representing four Dirac spinors lying in four different left ideals. A transformation R that is generated by bivectors $\gamma_\mu \wedge \gamma_\nu$, $\mu, \nu = 0, 1, 2, 3$, can act on $\psi^{\alpha i}$ from the left, in which case R has the role of a Lorentz transformation, or from the right, in which case it behaves as an ‘internal’ transformation. Our model thus incorporates the internal gauge group $SU(2)$. The covariant derivative occurring in the action contains a generalized spin connection that incorporates not only the ordinary spin connection but also two kinds of $SU(2)$ gauge fields, W_μ and W_μ^h , that are transformed into each other by the space inversion P . The ‘internal’ components of $\psi^{\alpha i}$, denoted by the index i that runs over four different left ideals, form two irreducible representations of $SU(2)$ that correspond to ordinary and mirror particles. Thus, we have found a theoretical explanation for the suggestion that there must be a mirror world—a parallel sector—with no common interaction with ordinary particles [2, 5]. In this paper, we demonstrate the existence of such a parallel sector for the case of weak interactions only.

2 Clifford algebra and spinors in Minkowski space

In constructing spinors, we will follow Ref. [17]. Let us consider the Minkowski space $M_{1,3}$ in which a quadratic form is $x^\mu \eta_{\mu\nu} x^\nu = x^2$, where $\eta_{\mu\nu} = \text{diag}(+ - - -)$ is the Minkowski metric tensor. The square root of x^2 is $x = x^\mu \gamma_\mu$, where γ_μ are generators of the Clifford algebra $Cl(1, 3)$ satisfying $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}$. An object $x^\mu \gamma_\mu$ is a vector in $M_{1,3}$.

Instead of the basis vectors γ_μ , $\mu = 0, 1, 2, 3$, we can introduce another set of basis

vectors (forming the so-called Witt basis),

$$\theta_1 = \frac{1}{2}(\gamma_0 + \gamma_3), \quad \theta_2 = \frac{1}{2}(\gamma_1 + i\gamma_2), \quad (1)$$

$$\bar{\theta}_1 = \frac{1}{2}(\gamma_0 - \gamma_3), \quad \bar{\theta}_2 = \frac{1}{2}(\gamma_1 - i\gamma_2), \quad (2)$$

which satisfy

$$\{\theta_a, \bar{\theta}_b\} = \eta_{ab}, \quad \{\theta_a, \theta_b\} = 0, \quad \{\bar{\theta}_a, \bar{\theta}_b\} = 0, \quad (3)$$

where $\eta_{ab} = \text{diag}(1, -1)$, $a, b = 1, 2$. Relations (3) are fermionic anticommutation relations.

Minkowski space $M_{1,3} \equiv M$ has been decomposed according to $M = N \dot{+} P$ into two singular subspaces with null quadratic forms.

We now observe that the product

$$f = \bar{\theta}_1 \bar{\theta}_2 \quad (4)$$

satisfies

$$\bar{\theta}_a f = 0, \quad a = 1, 2. \quad (5)$$

Therefore, f can be interpreted as ‘vacuum’, and $\bar{\theta}_a$ can be interpreted as operators that annihilate f . An object constructed as a superposition,

$$\Psi = (\psi^0 \underline{1} + \psi^1 \theta_1 + \psi^2 \theta_2 + \psi^{12} \theta_1 \theta_2) f, \quad (6)$$

is an element of a minimal left ideal of complexified $Cl(1, 3)$; it is a *spinor*. The notation of the components in Eq. (6) is adapted to the fact that Ψ is given in terms of a Clifford algebra of a 2-dimensional vector space spanned by θ_1, θ_2 acting on f . Because θ_1 and θ_2 are null vectors, satisfying $\theta_1^2 = 0, \theta_2^2 = 0$, the latter Clifford algebra is in fact the Grassmann algebra $\bigwedge N$.

Because Eq. (6) describes a 4-component spinor, it is convenient to change the notation according to

$$\Psi = (\psi^1 \underline{1} + \psi^2 \theta_1 \theta_2 + \psi^3 \theta_1 + \psi^4 \theta_2) f. \quad (7)$$

The even part of Ψ is a left-handed spinor, whilst the odd part is a right-handed spinor:

$$\Psi_L = (\psi^1 \underline{1} + \psi^2 \theta_1 \theta_2) \bar{\theta}_1 \bar{\theta}_2, \quad (8)$$

$$\Psi_R = (\psi^3 \theta_1 + \psi^4 \theta_2) \bar{\theta}_1 \bar{\theta}_2. \quad (9)$$

Defining $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$, we have

$$i\gamma_5 \Psi_L = -\Psi_L, \quad (10)$$

$$i\gamma_5 \Psi_R = \Psi_R. \quad (11)$$

To illustrate that Ψ indeed behaves as a spinor under rotation, let us consider as an example

$$R_{\gamma_1\gamma_2} = e^{\frac{1}{2}\gamma_1\gamma_2\varphi} = \cos \frac{\varphi}{2} + \gamma_1\gamma_2 \sin \frac{\varphi}{2}, \quad (12)$$

which (after using $\gamma_1 = \theta_2 + \bar{\theta}_2$, $\gamma_2 = (-i)(\theta_2 - \bar{\theta}_2)$ that follow from inverting Eqs. (1), (2)) becomes

$$R_{\gamma_1\gamma_2} = e^{\frac{i}{2}[\theta_2, \bar{\theta}_2]\varphi} = \cos \frac{\varphi}{2} + [\theta_2, \bar{\theta}_2] \sin \frac{\varphi}{2}. \quad (13)$$

Then, we find

$$R_{\gamma_1\gamma_2}\Psi = \left(e^{\frac{i\varphi}{2}}\psi^1\mathbf{1} + e^{-\frac{i\varphi}{2}}\psi^2\theta_1\theta_2 + e^{\frac{i\varphi}{2}}\psi^3\theta_2 + e^{-\frac{i\varphi}{2}}\psi^4\theta_2 \right) f, \quad (14)$$

where we have taken into account $\theta_a f = 0$. In Eq. (14), we recognize the well-known transformation of a 4-component spinor.

3 Four independent spinors

Thus far, we have considered one possible decomposition of $M_{1,3}$ into two singular subspaces, N and P , that are spanned, respectively, over the basis (θ_1, θ_2) and $(\bar{\theta}_1, \bar{\theta}_2)$. There is another possibility, namely, $(\theta_1, \bar{\theta}_2)$ and $(\bar{\theta}_1, \theta_2)$. We can thus consider four different vacua,

$$f_1 = \bar{\theta}_1\bar{\theta}_2, \quad f_2 = \theta_1\theta_2, \quad f_3 = \theta_1\bar{\theta}_2, \quad f_4 = \bar{\theta}_1\theta_2, \quad (15)$$

and can construct four different kinds of spinors, each being in a different minimal left ideal of $Cl(1, 3)$ (more precisely, of its complexified version):

$$\Psi^1 = (\psi^{11}\mathbf{1} + \psi^{21}\theta_1\theta_2 + \psi^{31}\theta_1 + \psi^{41}\theta_2)f_1, \quad (16)$$

$$\Psi^2 = (\psi^{12}\mathbf{1} + \psi^{22}\bar{\theta}_1\bar{\theta}_2 + \psi^{32}\bar{\theta}_1 + \psi^{42}\bar{\theta}_2)f_2, \quad (17)$$

$$\Psi^3 = (\psi^{13}\bar{\theta}_1 + \psi^{23}\theta_2 + \psi^{33}\mathbf{1} + \psi^{43}\bar{\theta}_1\theta_2)f_3, \quad (18)$$

$$\Psi^4 = (\psi^{14}\theta_1 + \psi^{24}\bar{\theta}_2 + \psi^{34}\mathbf{1} + \psi^{44}\theta_1\bar{\theta}_2)f_4. \quad (19)$$

An arbitrary element of $Cl(1, 3)$ is the sum of those independent spinors:

$$\Phi = \Psi^1 + \Psi^2 + \Psi^3 + \Psi^4 = \psi^{\alpha i} \xi_{\alpha i} \equiv \psi^{\tilde{A}} \xi_{\tilde{A}}, \quad (20)$$

where the set of elements $\{\mathbf{1}f_1, \theta_1\theta_2f_1, \dots, \theta_1f_4, \bar{\theta}_2f_4, \mathbf{1}f_4, \bar{\theta}_1\theta_2f_4\} \equiv \xi_{\tilde{A}} \equiv \xi_{\alpha i}$, $\alpha, i = 1, 2, 3, 4$, forms a spinor basis of $Cl(1, 3)$. We will call the object Φ a *generalized spinor*.

Spinors are thus just particular Clifford numbers. Usually a Clifford number is given in terms of 16 basis elements $\Gamma_M \equiv (\mathbf{1}, \gamma_\mu, \gamma_{\mu\nu}, \gamma_5\gamma_\mu, \gamma_5)$, but it can also be given in terms of the 16 basis elements $\xi_{\tilde{A}} \equiv \xi_{\alpha i}$, where the first index denotes four spinor components, whereas the second index denotes four independent left minimal ideals of $Cl(1, 3)$. In general, $\psi^{\alpha i}$ are complex-valued spacetime fields.

4 Behavior of spinors under discrete Lorentz transformations

We will now explore how the left-handed and right-handed spinors (8),(9) transform under space inversion P, under time reversal T, and under PT.

Space inversion, $\gamma_0 \rightarrow \gamma_0$, $\gamma_r \rightarrow -\gamma_r$, $r = 1, 2, 3$

$$\begin{aligned}\theta_1 &\rightarrow \frac{1}{2}(\gamma_0 - \gamma_3) = \bar{\theta}_1, \\ \theta_2 &\rightarrow \frac{1}{2}(-\gamma_1 - i\gamma_2) = -\theta_2, \\ \bar{\theta}_1 &\rightarrow \frac{1}{2}(\gamma_0 + \gamma_3) = \theta_1, \\ \bar{\theta}_2 &\rightarrow \frac{1}{2}(-\gamma_1 + i\gamma_2) = -\bar{\theta}_2.\end{aligned}\tag{21}$$

Using Eqs. (8),(9) and the above transformations of the Witt basis, we find

$$\Psi_L \rightarrow \Psi'_L = (-\psi^1 \underline{1} + \psi^2 \bar{\theta}_1 \theta_2) \theta_1 \bar{\theta}_2 \equiv \Psi_R^\natural \neq \Psi_R,\tag{22}$$

$$\Psi_R \rightarrow \Psi'_R = (-\psi^3 \bar{\theta}_1 + \psi^4 \theta_2) \theta_1 \bar{\theta}_2 \equiv \Psi_L^\natural \neq \Psi_L.\tag{23}$$

We see that the left-handed spinor Ψ_L does not transform into the right-handed Ψ_R of Eq. (9). More precisely, it does not transform into the right-handed spinor of the same ideal of $Cl(1, 3)$. Instead, it transforms into the spinor Ψ_R^\natural , which, according to Eq. (18), belongs to the third ideal. The transformed spinor Ψ_R^\natural is right-handed because it satisfies

$$i\gamma_5 \Psi_R^\natural = i\gamma_5 (-\psi^1 \underline{1} + \psi^2 \bar{\theta}_1 \theta_2) \theta_1 \bar{\theta}_2 = \Psi_R^\natural,\tag{24}$$

where we have used Eqs. (1),(2) and $\{\gamma_5, \gamma_\mu\} = 0$, from which it follows that $i\gamma_5 \theta_1 \bar{\theta}_2 = \theta_1 \bar{\theta}_2$.

Analogously, the right-handed spinor Ψ_R transforms into the spinor Ψ_L^\natural , which, according to Eq. (18), also belongs to the third ideal and satisfies

$$i\gamma_5 \Psi_L^\natural = i\gamma_5 (-\psi^3 \bar{\theta}_1 + \psi^4 \theta_2) \theta_1 \bar{\theta}_2 = -\Psi_L^\natural.\tag{25}$$

Therefore, Ψ_L^\natural is a left-handed spinor.

By including a new index j that runs over ideals, we have

$$\Psi^j = \Psi_L^j + \Psi_R^j, \quad j = 1, 2, 3, 4.\tag{26}$$

Under the space inversion, $\Psi_L^j \rightarrow \text{not } \Psi_R^j$, and $\Psi_R^j \rightarrow \text{not } \Psi_L^j$, but $\Psi_L^j \rightarrow \Psi_R^{j'}$, and $\Psi_R^j \rightarrow \Psi_L^{j'}$, where $j = 1, 2, 3, 4$ and $j' = 3, 4, 1, 2$. Thus, a left (right)-handed spinor of an ideal j transforms into a right (left)-handed spinor of a different ideal j' .

Time reversal, $\gamma_0 \rightarrow -\gamma_0$, $\gamma_r \rightarrow \gamma_r$, $r = 1, 2, 3$.

$$\begin{aligned}\theta_1 &\rightarrow \frac{1}{2}(-\gamma_0 + \gamma_3) = -\bar{\theta}_1, \\ \theta_2 &\rightarrow \frac{1}{2}(\gamma_1 + i\gamma_2) = \theta_2, \\ \bar{\theta}_1 &\rightarrow \frac{1}{2}(-\gamma_0 - \gamma_3) = -\theta_1, \\ \bar{\theta}_2 &\rightarrow \frac{1}{2}(\gamma_1 - i\gamma_2) = \bar{\theta}_2.\end{aligned}\tag{27}$$

By inserting the above transformations into Eqs. (8),(9), we obtain

$$\Psi_L \rightarrow \Psi_L'' = (-\psi^1 \underline{1} + \psi^2 \bar{\theta}_1 \theta_2) \theta_1 \bar{\theta}_2 = \Psi_R^\sharp, \tag{28}$$

$$\Psi_R \rightarrow \Psi_R'' = (\psi^3 \bar{\theta}_1 - \psi^4 \theta_2) \theta_1 \bar{\theta}_2 = -\Psi_L^\sharp. \tag{29}$$

Again, a left (right)-handed spinor transforms into a right (left)-handed spinor of a different ideal. In the case illustrated above, a left (right)-handed spinor of the first ideal transforms into a right (left)-handed spinor of the third ideal. In general, under time reversal, $\Psi_L^j \rightarrow \Psi_R^{j'}$ and $\Psi_R^j \rightarrow \Psi_L^{j'}$, where $j = 1, 2, 3, 4$ and $j' = 3, 4, 1, 2$.

Space inversion and time reversal, $\gamma_0 \rightarrow -\gamma_0$, $\gamma_r \rightarrow -\gamma_r$, $r = 1, 2, 3$.

$$\theta_a \rightarrow -\theta_a, \quad \bar{\theta}_a \rightarrow -\bar{\theta}_a, \quad a = 1, 2. \tag{30}$$

Then, we have $\Psi_L \rightarrow \Psi_L$ and $\Psi_R \rightarrow -\Psi_R$.

In general, under a Lorentz transformation, including a discrete (i.e., improper) one, an arbitrary element of $Cl(1, 3)$, Eq. (20), transforms according to

$$\Phi = \psi^{\tilde{A}} \xi_{\tilde{A}} \rightarrow \Phi' = \psi^{\tilde{A}} \xi'_{\tilde{A}} = \psi^{\tilde{A}} L_{\tilde{A}}^{\tilde{B}} \xi_{\tilde{B}} = \psi'^{\tilde{B}} \xi_{\tilde{B}}, \tag{31}$$

where

$$\xi'_{\tilde{A}} = L_{\tilde{A}}^{\tilde{B}} \xi_{\tilde{B}}, \tag{32}$$

$$\psi'^{\tilde{B}} = \psi^{\tilde{A}} L_{\tilde{A}}^{\tilde{B}}. \tag{33}$$

Here, $\tilde{A} = \alpha i$ and $\tilde{B} = \beta j$. Because \tilde{A} denotes the double index αi , we can arrange the basis elements $\xi_{\tilde{A}} \equiv \xi_{\alpha i}$ into a matrix as

$$\xi_{\tilde{A}} \equiv \xi_{\alpha i} = \begin{pmatrix} f_1 & f_2 & \bar{\theta}_1 f_3 & \theta_1 f_4 \\ \theta_1 \theta_2 f_1 & \bar{\theta}_1 \bar{\theta}_2 f_2 & \theta_2 f_3 & \bar{\theta}_2 f_4 \\ \theta_1 f_1 & \bar{\theta}_1 f_2 & f_3 & f_4 \\ \theta_2 f_1 & \bar{\theta}_2 f_2 & \bar{\theta}_1 \theta_2 f_3 & \theta_1 \bar{\theta}_2 f_4 \end{pmatrix} \tag{34}$$

and components $\psi^{\tilde{A}} \equiv \psi^{\alpha i}$ into a matrix as

$$\psi^{\alpha i} = \begin{pmatrix} \psi^{11} & \psi^{12} & \psi^{13} & \psi^{14} \\ \psi^{21} & \psi^{22} & \psi^{23} & \psi^{24} \\ \psi^{31} & \psi^{32} & \psi^{33} & \psi^{34} \\ \psi^{41} & \psi^{42} & \psi^{43} & \psi^{44} \end{pmatrix}. \tag{35}$$

For instance, under space inversion (21), the Witt basis elements transform according to (21), and the basis elements (34) transform according to (2.13) into new basis elements,

$$\xi'_{\alpha i} = \begin{pmatrix} -f_3 & -f_4 & -\theta_1 f_1 & -\bar{\theta}_1 f_2 \\ \bar{\theta}_1 \theta_2 f_3 & \theta_1 \bar{\theta}_2 f_4 & \theta_2 f_1 & \bar{\theta}_2 f_2 \\ -\bar{\theta}_1 f_3 & -\theta_1 f_4 & -f_1 & -f_2 \\ \theta_2 f_3 & \bar{\theta}_2 f_4 & \theta_1 \theta_2 f_1 & \bar{\theta}_1 \bar{\theta}_2 f_2 \end{pmatrix}, \quad (36)$$

whereas the components transform according to (33) as

$$\psi^{\alpha i} = \begin{pmatrix} -\psi^{33} & -\psi^{34} & -\psi^{31} & -\psi^{32} \\ \psi^{43} & \psi^{44} & \psi^{41} & \psi^{42} \\ -\psi^{13} & -\psi^{14} & -\psi^{11} & -\psi^{12} \\ \psi^{23} & \psi^{24} & \psi^{21} & \psi^{22} \end{pmatrix}. \quad (37)$$

Because we consider the active transformations, then, according to (31), either Eq. (32) or Eq. (33) holds, but both cannot hold at once. A generic Clifford number Φ of (20), a sum of the spinors of four different left ideals, thus transforms into a different object Φ' that is expanded, according to Eq. (31), either in terms of a new basis (36) and old components or in terms of the old basis and new components (37).

5 An action for the generalized spinor field

A possible action for a spacetime-dependent field $\psi^{\tilde{A}}$ is¹

$$I[\psi^*_{\tilde{A}}, \psi^{\tilde{B}}] = \int d^4x \sqrt{-g} i \psi^*_{\tilde{A}} (\gamma^\mu)^{\tilde{A}}_{\tilde{B}} D_\mu \psi^{\tilde{B}}. \quad (38)$$

Here [14, 15],

$$(\gamma^\mu)^{\tilde{A}}_{\tilde{B}} = \langle \xi^{\tilde{A}\dagger} \gamma^\mu \xi_{\tilde{B}} \rangle_S = \delta^i_j (\gamma^\mu)^\alpha_\beta \quad \alpha, \beta, \gamma, \delta = 1, 2, 3, 4 \quad (39)$$

are matrix elements of the 16-dimensional, reducible, representation of the generators γ^μ . The operation $\langle A \rangle_S$ takes the scalar part of a generic Clifford number A and multiplies the result by the dimension $n = 4$ of the minimal left ideal (i.e., the dimension of a spinor of $Cl(1, 3)$). The symbol ‘ \ddagger ’ denotes reversion, i.e., the operation that reverses the order of all 1-vectors in an expression.

Indices are lowered and raised by the metric $Z_{\tilde{A}\tilde{B}} = \langle \xi^{\tilde{A}\dagger} \xi_{\tilde{B}} \rangle_S = z_{ij} z_{\alpha\beta}$ and its inverse $Z^{\tilde{A}\tilde{B}}$.

The part of action (38) that belongs to one particular left ideal is obtained by fixing the ideal index, e.g., $i = j = 1$ in Eq.(39), and writing $\psi^*_{\alpha 1} \equiv \psi_\alpha$, $\psi^{\beta 1} \equiv \psi^\beta$:

$$I[\psi^*_\alpha, \psi^\beta] = \int d^4x \sqrt{-g} i \psi^*_\alpha (\gamma^\mu)^\alpha_\beta D_\mu \psi^\beta. \quad (40)$$

¹This action can be embedded into an even more general action that depends on position in Clifford space, a manifold whose tangent space at any point is a Clifford algebra [14, 15].

Here, $(\gamma^\mu)^\alpha_\beta$ are the ordinary Dirac matrices satisfying $(\gamma^\mu)^\alpha_\gamma(\gamma^\nu)^\gamma_\beta + (\gamma^\nu)^\alpha_\gamma(\gamma^\mu)^\gamma_\beta = 2\eta^{\mu\nu}\delta^\alpha_\beta$. A complex conjugate spinor with the lower index α can be expressed in terms of a spinor with the upper index α as $\psi^*_\alpha = z_{\alpha\beta}\psi^{*\beta}$. Because, in particular classes of representations, $z_{\alpha\beta}$ equal $(\gamma^0)^\alpha_\beta$, we have that ψ^*_α corresponds to $\bar{\psi} = \psi^\dagger\gamma^0$.

The covariant derivative in (38) reads

$$D_\mu\psi^{\tilde{A}} = \partial_\mu\psi^{\tilde{A}} + G_\mu^{\tilde{A}}{}_{\tilde{B}}\psi^{\tilde{B}}, \quad (41)$$

where $G_\mu^{\tilde{A}}{}_{\tilde{B}} \equiv G_\mu^{\alpha i}{}_{\beta j}$ is a gauge field that, in general, couples to the spinors of different left ideals (denoted by indices i or j).

The interactive term Lagrangian in the action (38) can be written explicitly as

$$\mathcal{L}_{\text{int}} = \psi^*_{\alpha i}(\gamma^\mu)^{\alpha i}{}_{\beta j}G_\mu^{\beta j}{}_{\rho k}\psi^{\rho k} = \psi^*_{\alpha i}(\gamma^\mu)^\alpha_\beta G_\mu^{\beta i}{}_{\rho k}\psi^{\rho k}. \quad (42)$$

If acting within the same left ideal, the field $G_\mu^{\beta i}{}_{\rho k}$ for $i = k$ behaves as the ordinary spin connection in curved spacetime. Otherwise, it behaves as a gauge field due to the “internal” local gauge transformations [14, 15] that transform one ideal into the other. The group of the latter transformations contain SU(2) as a subgroup.

Let us write Eq. (42) in a more compact form,

$$\mathcal{L}_{\text{int}} = \bar{\psi}_i \gamma^\mu G_\mu^i{}_k \psi^k, \quad (43)$$

where all quantities now contain the implicit spinor indices α, β . In the chiral representation,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \sigma^\mu = (1, \sigma^r), \quad \bar{\sigma}^\mu = (1, -\sigma^r), \quad r = 1, 2, 3, \quad (44)$$

where σ^r are the Pauli 2×2 matrices. If we write the gauge fields in the following block matrix form,

$$G_\mu^i{}_k = \begin{pmatrix} a_\mu^i{}_k & b_\mu^i{}_k \\ c_\mu^i{}_k & d_\mu^i{}_k \end{pmatrix}, \quad (45)$$

then

$$\gamma^\mu G_\mu^i{}_k \psi^k = \begin{pmatrix} \sigma^\mu c_\mu^i{}_k & \sigma^\mu d_\mu^i{}_k \\ \bar{\sigma}^\mu a_\mu^i{}_k & \bar{\sigma}^\mu b_\mu^i{}_k \end{pmatrix} \begin{pmatrix} \chi_L^k \\ \chi_R^k \end{pmatrix}. \quad (46)$$

Here, $a_\mu^i{}_k, b_\mu^i{}_k, c_\mu^i{}_k, d_\mu^i{}_k$ are 2×2 matrices, whilst χ_L^k, χ_R^k are 2-component Weyl spinors. Because $\bar{\psi}_i = (\chi_{iR}^\dagger, \chi_{iL}^\dagger)$, we have

$$\mathcal{L}_{\text{int}} = \chi_{iR}^\dagger \sigma^\mu c_\mu^i{}_k \chi_L^k + \chi_{iR}^\dagger \sigma^\mu d_\mu^i{}_k \chi_R^k + \chi_{iL}^\dagger \bar{\sigma}^\mu a_\mu^i{}_k \chi_L^k + \chi_{iL}^\dagger \bar{\sigma}^\mu b_\mu^i{}_k \chi_R^k. \quad (47)$$

This interactive Lagrangian has an interesting structure that involves spinor degrees of freedom and “internal” degrees of freedom associated with different left ideals (denoted by indices i, k) and various gauge fields. Left-handed or right-handed spinors of one left

ideal, denoted by k , are coupled (via various gauge fields) to the corresponding spinors of another left ideal, denoted by i (where, in particular, it can be $i = k$).

If we split the indices according to $i = \bar{i}, \underline{i}$ and $k = \bar{k}, \underline{k}$, where $\bar{i}, \bar{k} = 1, 2$ and $\underline{i}, \underline{k} = 3, 4$, then the Lagrangian (47) can be written in the form of

$$\begin{aligned} \mathcal{L}_{\text{int}} = & \chi_{iR}^\dagger \sigma^\mu c_{\mu \bar{k}} \chi_L^{\bar{k}} + \chi_{iL}^\dagger \bar{\sigma}^\mu b_{\mu \bar{k}} \chi_R^{\bar{k}} + \chi_{iR}^\dagger \sigma^\mu c_{\mu \underline{k}} \chi_L^{\underline{k}} + \chi_{iL}^\dagger \bar{\sigma}^\mu b_{\mu \underline{k}} \chi_R^{\underline{k}} \\ & + \chi_{iL}^\dagger \bar{\sigma}^\mu a_{\mu \bar{k}} \chi_L^{\bar{k}} + \chi_{iR}^\dagger \sigma^\mu d_{\mu \bar{k}} \chi_R^{\bar{k}} + \chi_{iL}^\dagger \bar{\sigma}^\mu a_{\mu \underline{k}} \chi_L^{\underline{k}} + \chi_{iR}^\dagger \sigma^\mu d_{\mu \underline{k}} \chi_R^{\underline{k}}. \end{aligned} \quad (48)$$

The bared indices denote the 1st and 2nd ideals, whereas the underlined indices denote the 3rd and 4th ideals.

One can show that a rotation from the right, $\Phi \rightarrow \Phi' = \Phi R_{\gamma_r \gamma_s}$, in general transforms the 1st into the 2nd and the 3rd into the 4th ideal. The left-handed field $\chi_L^{\bar{k}}$, $\bar{k} = 1, 2$ forms the fundamental representation of SU(2), and it is coupled via the SU(2) gauge field $a_{\mu \bar{k}}$ to χ_{iL}^\dagger and via $c_{\mu \bar{k}}$ to χ_{iR}^\dagger . The analogous holds for the fields $\chi_L^{\underline{k}}$, $\chi_R^{\bar{k}}$, and $\chi_R^{\underline{k}}$.

We have seen in Eqs. (36),(37) that space inversion transforms the 1st ideal into the 3rd one and the 2nd ideal into the 4th one. Besides that, space inversion exchanges the upper half of the 4-component spinor with the lower half, i.e., it exchanges $\chi_L^{\underline{k}} \rightarrow \chi_R^{k'}$ and $\chi_L^{\bar{k}} \leftarrow \chi_R^{k'}$. Instead of the inversion (21), it is convenient to apply the reflection $\gamma_0 \rightarrow \gamma_0$, $\gamma_1 \rightarrow \gamma_1$, $\gamma_2 \rightarrow \gamma_2$, $\gamma_3 \rightarrow -\gamma_3$, because then no changes of sign take place in the transformations of $\xi_{\alpha i}$, $\psi^{\alpha i}$, and $G^{\alpha i}_{\beta j}$. Using (32), one can show that

$$G_{\mu \beta j}^{\alpha i} \rightarrow G_{\mu \beta j}'^{\alpha i} = L_{\delta k}^{\alpha i} L^{\gamma \ell}_{\beta j} G_{\mu \gamma \ell}^{\delta k}. \quad (49)$$

So, we find that

$$a_{\mu \bar{k}} \rightarrow d_{\mu \underline{k}}, \quad a_{\mu \underline{k}} \rightarrow d_{\mu \bar{k}}, \quad (50)$$

$$c_{\mu \bar{k}} \rightarrow b_{\mu \underline{k}}, \quad b_{\mu \underline{k}} \rightarrow c_{\mu \bar{k}}. \quad (51)$$

The interaction Lagrangian (47) (which can be written in the form of (48)) is thus invariant under space inversion. A notorious feature of this Lagrangian is that an ordinary right-handed field $\chi_R^{\bar{k}}$, $\bar{k} = 1, 2$, is not coupled to the same gauge field as the left-handed field $\chi_L^{\bar{k}}$. The right-handed field $\chi_R^{\bar{k}}$ behaves as a “singlet” with respect to the SU(2) gauge field $a_{\mu \bar{k}}$, which the left-handed field $\chi_L^{\bar{k}}$ is coupled to. Instead, $\chi_R^{\bar{k}}$ behaves as an SU(2) doublet with respect to another gauge field, namely, $d_{\mu \bar{k}}$.

Lagrangian (48) contains, amongst others, a term $\chi_{iL}^\dagger \bar{\sigma}^\mu a_{\mu \bar{k}} \chi_L^{\bar{k}}$. As already mentioned, $a_{\mu \bar{k}}$ is a 2×2 matrix in the spinor indices $\bar{\alpha}$, $\bar{\beta} = 1, 2$. If, in particular, $a_{\mu \bar{k}}$ is a diagonal matrix in indices $\bar{\alpha}$, $\bar{\beta}$, then the term $\chi_{iL}^\dagger \bar{\sigma}^\mu a_{\mu \bar{k}} \chi_L^{\bar{k}}$ is just like the interactive Lagrangian for weak interactions, $a_{\mu \bar{k}}$ being the weak SU(2) gauge fields $W_{\mu \bar{k}}$.

However, Lagrangian (48) contains other terms besides the one describing the usual weak interaction. It is beyond the scope of this Letter to investigate the physical meaning of those extra terms. One possibility is that due to certain superselection rules, the fields

$c_{\mu \underline{k}}^{\bar{i}}$ and $b_{\mu \underline{k}}^{\bar{i}}$ vanish. Another possibility is that Eq. (48) predicts the existence of new interactions. From now on, we will only consider the part of \mathcal{L}_{int} that contains the fields $a_{\mu \underline{k}}^{\bar{i}}$ and $d_{\mu \underline{k}}^{\bar{i}}$.

We adopt an interpretation that the group $\text{SU}(2)$ of $R_{\gamma_r \gamma_s} \in \text{SU}(2)$ that acts on a generalized spinor Φ from the right and that “mixes” either indices \bar{i}, \bar{k} or $\underline{i}, \underline{k}$ is the gauge group of weak interactions. Because we know from experiments that right-handed ordinary fields do not interact under the weak $\text{SU}(2)$ gauge fields, we will assume $d_{\mu \underline{k}}^{\bar{i}} = 0$. In addition, let us also assume $a_{\mu \underline{k}}^{\bar{i}} = 0$. Then, the Lagrangian (48) contains the following part:

$$\mathcal{L}_{\text{int}}^W = \chi_{\bar{i}L}^\dagger \bar{\sigma}^\mu a_{\mu \bar{k}}^{\bar{i}} \chi_L^{\bar{k}} + \chi_{\underline{i}R}^\dagger \sigma^\mu d_{\mu \underline{k}}^{\bar{i}} \chi_R^{\underline{k}}, \quad \bar{i}, \bar{k} = 1, 2, \quad \underline{i}, \underline{k} = 3, 4. \quad (52)$$

Here, we can adopt an interpretation that $\chi_L^{\bar{k}}$ are spinor fields describing ordinary particles coupled to the ordinary weak $\text{SU}(2)$ gauge fields, whereas $\chi_R^{\underline{k}}$ are *mirror* fields describing mirror particles that are coupled to the mirror gauge fields $d_{\mu \underline{k}}^{\bar{i}}$. Notice the difference between the barred and underlined indices. In other words, spinor fields of the 1st and 2nd ideals describe ordinary particles (and antiparticles), whereas spinor fields of the 3rd and 4th ideals describe mirror particles (and mirror antiparticles).

Putting it into a more compact notation, we write (52) as

$$\mathcal{L}_{\text{int}}^W = \chi_L^\dagger \bar{\sigma}^\mu W_\mu \chi_L + \chi_R^{\natural\dagger} \sigma^\mu W_\mu^{\natural} \chi_R^{\natural}, \quad (53)$$

where ‘ \natural ’ denotes mirror fields, so that $\chi_L \equiv \chi_L^{\bar{k}}$ and $\chi_R^{\natural} \equiv \chi_R^{\underline{k}}$. Here, W_μ and W_μ^{\natural} are $\text{SU}(2)$ gauge fields, and thus they are 2×2 matrices in indices \bar{i}, \bar{k} and $\underline{i}, \underline{k}$, respectively, i.e., $W_\mu \equiv a_{\mu \bar{k}}^{\bar{i}}$ and $W_\mu^{\natural} \equiv d_{\mu \underline{k}}^{\bar{i}}$.

The action (38) then contains the following part:

$$I^W = \int d^4x \sqrt{-g} i \left(\bar{\psi}_L \gamma^\mu \partial_\mu \psi_L + \bar{\psi}_L \gamma^\mu W_\mu \psi_L + \bar{\psi}_R^{\natural} \gamma^\mu \partial_\mu \psi_R^{\natural} + \bar{\psi}_R^{\natural} \gamma^\mu W_\mu^{\natural} \psi_R^{\natural} \right), \quad (54)$$

where

$$\psi_L = \frac{1}{2}(1 - i\gamma_5)\psi = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix} \equiv \begin{pmatrix} \chi_L^{\bar{k}} \\ 0 \end{pmatrix} \quad (55)$$

$$\bar{\psi}_L = (0, \chi_L^\dagger) \equiv (0, \chi_L^{\bar{k}\dagger}) \quad (56)$$

$$\psi_R^{\natural} = \frac{1}{2}(1 + i\gamma_5)\psi = \begin{pmatrix} 0 \\ \chi_R^{\natural} \end{pmatrix} \equiv \begin{pmatrix} 0 \\ \chi_R^{\underline{k}} \end{pmatrix} \quad (57)$$

$$\bar{\psi}_R^{\natural} = (\chi_R^{\natural\dagger}, 0) \equiv (\chi_R^{\underline{k}\dagger}, 0). \quad (58)$$

In Eq. (53), we identified the weak gauge field W_μ with the diagonal part (with respect to hidden indices $\bar{\alpha}, \bar{\beta}$) of $a_{\mu \bar{k}}^{\bar{i}}$ and the mirror gauge field W_μ^{\natural} with the diagonal part of $d_{\mu \underline{k}}^{\bar{i}}$. Remember that the latter gauge fields, in addition to being matrices in the “internal” indices \bar{i}, \bar{k} and $\underline{i}, \underline{k}$, are also matrices in the spinor indices $\bar{\alpha}, \bar{\beta} = 1, 2$ (that, for simplicity,

are not explicitly displayed). On the other hand, if we take $i = k$ and allow for $\bar{\alpha} \neq \bar{\beta}$, then $a_{\mu}^i{}_k$ and $d_{\mu}^i{}_k$ correspond to the ordinary spin connections².

The action (54), which besides the ordinary particles described by ψ_L and the ordinary gauge fields W_{μ} contains the corresponding mirror partners ψ_R^{\dagger} and W_{μ}^{\dagger} , encodes the proposal by Kobzarev et al. [2]. Later, a model essentially equivalent to that given by the action (54) was proposed by Foot et al. [5]. In this paper, we have shown how such a system with ordinary and mirror particles coupled to the corresponding gauge fields can be described within a framework based on the algebraic spinors and their behavior under space inversion.

6 Discussion and conclusion

The observation that space inversion P and time reversal T transform a spinor of one left ideal of $Cl(1, 3)$ into a spinor of another left ideal brings a novel insight on the invariance of nature under P and T that, to my knowledge, has not yet been explicitly pointed out. The idea that the invariance of weak interactions under P and T can be restored by introducing mirror particles has been considered by Lee and Yang [1] and later in Refs. [2]–[5]. In this letter, we provide a theoretical basis for such a model that comes naturally from the well-known theory of algebraic spinors, i.e., the objects embedded within the Clifford algebra that operates on them.

Electron spin was first postulated based on experimental evidence in atomic spectra and was later explained by the Dirac equation, which provided a deep theoretical insight. We have a somewhat analogous situation with mirror particles that were postulated to restore left-right symmetry. In this paper, we have shown that the generalized spinors, which encompass all four left ideals, describe not only the ordinary spin and weak isospin but also the ordinary and mirror particles. The corresponding SU(2) gauge fields are incorporated in the generalized spin connection. One further step that should be done is to include the U(1) gauge field and to show precisely how the electric charge and the electromagnetic field occur within this Clifford algebra-based theory. To include the color SU(3) as well, an extension of $Cl(1, 3)$ or its complex version $Cl(4, \mathbb{C})$ to a suitable higher dimensional Clifford algebra, such as $Cl(8, \mathbb{C})$, would be necessary.

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²This can be shown by observing that for $i = k$, the generalized spin connection $G_{\mu}^{\alpha i}{}_{\beta k}$ is just the ordinary spin connection $\Gamma_{\mu}^{\alpha}{}_{\beta} = \frac{1}{2}\omega_{\mu}^{ab}[\gamma_a, \gamma_b]$; then, by taking into account Eqs. (44), one finds that in Eq. (46), the part with $a_{\mu}^i{}_k$ and $d_{\mu}^i{}_k$, for $i = k$, corresponds to the contribution of the ordinary spin connection.

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